HEAT AND MOISTURE TRANSFER IN LAMINA UNDER GENERALIZED BOUNDARY CONDITIONS

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An analytic solution is obtained for a problem in the theory of heat and mass transfer under generalized boundary conditions.

The analytic theory of heat and mass transfer was developed in detail by Lykov and others in [1,2].

In [3] the temperature and moisture distribution was studied in a capillary-porous lamina for zero mass flow and for given temperature on one side and given boundary conditions of the third kind on the other side of the lamina. A similar problem was solved in [4], the one difference being that on the first surface the heat flow was given instead of the temperature. The problem is of considerable interest in the theory of drying; it can also be found in the book of Krasnikov [5]. By using the results of [6] and [7] the exact solution of the problem was obtained and it was shown by numerical examples that the system of equations analyzed in [3] may lead to considerable errors.

The problems formulated in [3] and [4] are very special cases of the generalized Prudnikov problem [8]; it was solved by him by using the Datsev approach [9] regarding, first of all, the potentials on both sides of the lamina as given time functions. The system of Volterra integral equations of the first kind was then used to determine these potentials. These equations can, in turn, be reduced to generalized Abel integral equations and also to a system Volterra integral equations of the second kind.

In the present article an exact analytic solution is given for the generalized Prudnikov problem [8] which one can easily program for a digital computer. The solution is sought of the Lykov system:

$$\frac{\partial\theta_1(X,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}}=\frac{\partial^2\theta_1(X,\ \mathrm{Fo})}{\partial X^2}-\mathrm{Ko}^*\frac{\partial\theta_2(X,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}}, \tag{1}$$

$$\frac{\partial\theta_2(X,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}}=\mathrm{Lu}\frac{\partial^2\theta_2(X,\ \mathrm{Fo})}{\partial X^2}-\mathrm{Lu\,Pn}\frac{\partial^2\theta_1(X,\ \mathrm{Fo})}{\partial X^2} \tag{2}$$

under the initial conditions

$$\theta_k(X,\ 0)=0,\quad k=1,\ 2 \tag{3}$$

and the boundary conditions

$$\sum_{k=1}^{2}\left\{K_{m,k}\theta_k(0,\ \mathrm{Fo})+K_{m,k+2}\frac{\partial\theta_k(0,\ \mathrm{Fo})}{\partial X}\right\}=\varphi_m(\mathrm{Fo}),\quad m=1,\ 2, \tag{4}$$

$$\sum_{k=1}^{2}\left\{K_{m+2,k}\theta_k(1,\ \mathrm{Fo})+K_{m+2,k+2}\frac{\partial\theta_k(1,\ \mathrm{Fo})}{\partial X}\right\}=\varphi_{m+2}(\mathrm{Fo}),\quad m=1,\ 2 \tag{5}$$

where $K_{m,k}$ is the totality of similarity criteria maintaining a constant value during the entire heat and mass transfer.

By applying the Laplace transform to the system (1)-(2) together with the initial conditions (3), solutions were obtained in [10] in the operator form [p. 117, Eqs. (4 — 2 — 5) and (4 — 2 — 6)] which we now rewrite in a more suitable form:

Sofia, Bulgaria. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 31, No. 2, pp. 347-350, August, 1976. Original article submitted December 27, 1973.

$$\bar{\theta}_k(X, s) = \sum_{j=1}^{2} \left(\frac{\vartheta_j^2 - 1}{\mathrm{Ko}^*}\right)^{k-1} \{C_j \cos(\vartheta_j i \sqrt{s}\, X) + C_{j+2} \sin(\vartheta_j i \sqrt{s}\, X)\},$$
$$k = 1, 2, \tag{6}$$

where

$$\vartheta_j^2 = 1/2\{1 + \mathrm{Ko}^*\,\mathrm{Pn} + 1/\mathrm{Lu} + (-1)^j \sqrt{(1 + \mathrm{Ko}^*\,\mathrm{Pn} + 1/\mathrm{Lu})^2 - 4/\mathrm{Lu}}\},$$
$$j = 1, 2. \tag{7}$$

The new constants C_j and C_{j+2} are now determined from the condition that the transform must satisfy the boundary conditions

$$\sum_{j=1}^{2} \{\alpha_{mj} C_j + \beta_{mj} \vartheta_j i \sqrt{s}\, C_{j+2}\} = \bar{\varphi}_m(s), \quad m = 1, 2, \tag{8}$$

$$\sum_{j=1}^{2} \{A_{mj}(i\sqrt{s})\, C_j + B_{mj}(i\sqrt{s})\, \vartheta_j i \sqrt{s}\, C_{j+2}\} = \bar{\varphi}_{m+2}(s), \quad m = 1, 2, \tag{9}$$

where

$$\alpha_{mj} = K_{m,1} + \frac{\vartheta_j^2 - 1}{\mathrm{Ko}^*} K_{m,2}, \quad \beta_{mj} = K_{m,3} + \frac{\vartheta_j^2 - 1}{\mathrm{Ko}^*} K_{m,4}, \tag{10}$$

$$A_{mj}(i\sqrt{s}) = \alpha_{mj} \cos(\vartheta_j i \sqrt{s}) - \beta_{mj} \vartheta_j i \sqrt{s} \sin(\vartheta_j i \sqrt{s}), \tag{11}$$

$$B_{mj}(i\sqrt{s}) = \alpha_{mj} \frac{\sin(\vartheta_j i \sqrt{s})}{\vartheta_j i \sqrt{s}} + \beta_{mj} \cos(\vartheta_j i \sqrt{s}). \tag{12}$$

The case of constant external forces φ_m and φ_{m+2} will be considered below. Having determined C_j and C_{j+2} from (8) and (9), the solution (6) is rewritten as follows:

$$\bar{\theta}_k(X, s) = \frac{1}{sD(i\sqrt{s})} \sum_{j=1}^{2} \left(\frac{\vartheta_j^2 - 1}{\mathrm{Ko}^*}\right)^{k-1} \left\{D_j(i\sqrt{s}) \cos(\vartheta_j i \sqrt{s}\, X) + D_{j+2}(i\sqrt{s}) \frac{\sin(\vartheta_j i \sqrt{s}\, X)}{\vartheta_j i \sqrt{s}}\right\}, \tag{13}$$

where

$$D(i\sqrt{s}) = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ A_{31}(i\sqrt{s}) & A_{32}(i\sqrt{s}) & B_{31}(i\sqrt{s}) & B_{32}(i\sqrt{s}) \\ A_{41}(i\sqrt{s}) & A_{42}(i\sqrt{s}) & B_{41}(i\sqrt{s}) & B_{42}(i\sqrt{s}) \end{vmatrix}, \tag{14}$$

and $D_j(i\sqrt{s})$ is the determinant obtained from $D(i\sqrt{s})$ by replacing the j-th column by the right-hand sides of the system (8)-(9) in the case of the constants φ_m and φ_{m+2}.

To be able to go back to the originals one can apply the expansion theorem [10], since the solution (13) represents a ratio of two generalized polynomials. The common denominator $sD(i\sqrt{s})$ has $s = 0$, a zero root, and $s = s_n$, an infinite set of roots, which satisfy the characteristic equation

$$D(\mu_n) = 0, \tag{15}$$

where $\mu_n = i\sqrt{s_n}$.

Having carried out the necessary calculations, one obtains for the nonstationary potential distribution

$$\theta_k(X, \mathrm{Fo}) = \frac{1}{D(0)} \sum_{j=1}^{2} \left(\frac{\vartheta_j^2 - 1}{\mathrm{Ko}^*}\right)^{k-1} \{D_j(0) + D_{j+2}(0)\, X\} -$$
$$- \sum_{n=1}^{\infty} \exp(-\mu_n^2 \mathrm{Fo}) \sum_{j=1}^{2} \left(\frac{\vartheta_j^2 - 1}{\mathrm{Ko}^*}\right)^{k-1} \frac{2}{\mu_n \psi(\mu_n)} \left\{D_j(\mu_n) \cos(\vartheta_j \mu_n X) + D_{j+2}(\mu_n) \frac{\sin(\vartheta_j \mu_n X)}{\vartheta_j \mu_n X}\right\}, \quad k = 1, 2, \tag{16}$$

where

$$\psi(\mu_n) = \vartheta_1^2 \{D_{p1}(\mu_n) + D_{p3}(\mu_n)\} + \vartheta_2^2 \{D_{p2}(\mu_n) + D_{p4}(\mu_n)\}, \tag{17}$$

and $D_{pj}(\mu_n)$ is the determinant obtained from $D(\mu_n)$ by replacing the j-th column by the column $\{0, 0, P_{3j}(\mu_n), P_{4j}(\mu_n)\}$.

The expressions needed for computing $D_j(0)$, $D_j(\mu_n)$, and $D_{pj}(\mu_n)$ are given by

$$A_{mj}(0)=\alpha_{mj},\quad A_{mj}(\mu_n)=\alpha_{mj}\cos(\vartheta_j\mu_n)-\beta_{mj}\vartheta_j\mu_n\sin(\vartheta_j\mu_n), \tag{18}$$

$$B_{mj}(0)=\alpha_{mj}+\beta_{mj},\quad B_{mj}(\mu_n)=\alpha_{mj}\frac{\sin(\vartheta_j\mu_n)}{\vartheta_j\mu_n}+\beta_{mj}\cos(\vartheta_j\mu_n), \tag{19}$$

$$P_{mj}(\mu_n)=(\alpha_{mj}+\beta_{mj})\frac{\sin(\vartheta_j\mu_n)}{\vartheta_j\mu_n}+\dot{\beta}_{mj}\cos(\vartheta_j\mu_n), \tag{20}$$

$$P_{m,j+2}(\mu_n)=\left(\frac{\alpha_{mj}}{\vartheta_j^2\mu_n^2}+\beta_{mj}\right)\frac{\sin(\vartheta_j\mu_n)}{\vartheta_j\mu_n}-\frac{\alpha_{mj}}{\vartheta_j^2\mu_n^2}\cos(\vartheta_j\mu_n). \tag{21}$$

The obtained solution (16) can only be used in the case of $D(0) \neq 0$ and only for the constants φ_m and φ_{m+2}. In the case of the external potentials $\varphi_m(Fo)$ and $\varphi_{m+2}(Fo)$ being given as functions of time the solution must be transformed similarly as in [11].

One finds that it is easy to program the solution (16) for a digital computer all the more since there are subprograms available for the evaluation of determinants. An attempt by the author of [4] to solve a particular case of the problem under consideration by using Laplace transforms proved unsuccessful, since the expanding of the determinants led to some very cumbersome expressions.

NOTATION

X, dimensionless coordinate; Fo, Fourier number; θ_1, dimensionless temperature; θ_2, dimensionless potential of transfer of matter; Ko*, modified Kossovich criterion; Lu, Lykov criterion; Pn, Posnov criterion; $K_{m,k}$, totality of similarity criteria; φ_m, φ_{m+2}, external potentials.

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